METABELIAN SL$(n, \mathbb{C})$ REPRESENTATIONS OF KNOT GROUPS II: FIXED POINTS AND DEFORMATIONS

HANS U. BODEN AND STEFAN FRIEDL

ABSTRACT. Given a knot K in an integral homology sphere Σ with exterior N_K, there is a natural action of the cyclic group $\mathbb{Z}/n$ on the space of SL$(n, \mathbb{C})$ representations of the knot group $\pi_1(N_K)$, and this induces an action on the SL$(n, \mathbb{C})$ character variety $X_n(N_K)$. We identify the fixed points of this action in $X_n^*(N_K)$ with characters of irreducible metabelian representations.

We then show that for any irreducible metabelian representation α, we have $\dim_{\mathbb{C}} H^1(N_K; sl(n, \mathbb{C})_{ad\,\alpha}) \geq n - 1$. If equality holds, then we prove that the character ξ_α is a smooth point in $X_n(N_K)$ and that there exists an $(n-1)$–dimensional family of characters of irreducible SL$(n, \mathbb{C})$ representations of $\pi_1(N_K)$ near ξ_α. We relate the cohomological condition that $\dim_{\mathbb{C}} H^1(N_K; sl(n, \mathbb{C})_{ad\,\alpha}) = n - 1$ to the vanishing of $H_1(\widehat{\Sigma}_\varphi)$, where $\widehat{\Sigma}_\varphi$ denotes the associated metabelian branched cover of Σ branched along K.

1. INTRODUCTION

Suppose K is a knot. Throughout this paper we will always understand this to mean that K is an oriented simple closed curve in an integral homology 3-sphere Σ. We write $N_K = \Sigma^3 \smallsetminus \tau(K)$, where $\tau(K)$ denotes an open tubular neighborhood of K. In a previous paper ([BF08]) the authors classified the metabelian SL$(n, \mathbb{C})$ representations from the knot group $\pi_1(N_K)$. In this paper we continue our study of metabelian representations of knot groups.

In order to state our results, we need to introduce some terminology.

Given a topological space M, let $R_n(M)$ be the space of SL$(n, \mathbb{C})$ representations of $\pi_1(M)$ and $X_n(M)$ the associated character variety. We use ξ_α to denote the character of the representation $\alpha \colon \pi_1(M) \to$ SL$(n, \mathbb{C})$, and we recall the elementary but important fact that two irreducible representations determine the same character if and only if they are conjugate (see [LM85, Corollary 1.33]).

Now suppose K is a knot. There is an action of the group $\mathbb{Z}/n$ on the representation variety $R_n(N_K)$ given by twisting by the n–th roots of unity $\omega^k = e^{2\pi i k/n} \in U(1)$.

(This is a special case of the more general twisting operation described in [LM85, Ch. 5].) More precisely, we write $\mathbb{Z}/n = \langle \sigma \mid \sigma^n = 1 \rangle$ and set $(\sigma \cdot \alpha)(g) = \omega^{\varepsilon(g)}\alpha(g)$ for each $g \in \pi_1(N_K)$, where $\varepsilon \colon \pi_1(N_K) \to H_1(N_K) = \mathbb{Z}$ is determined by the given orientation of the knot.

This constructs an action of $\mathbb{Z}/n$ on $R_n(N_K)$ which, in turn, determines a corresponding action on the character variety $X_n(N_K)$. Our first result identifies the fixed points of $\mathbb{Z}/n$ in $X_n^*(N_K)$, the irreducible characters, as those associated to metabelian representations.

Theorem 1. *The character ξ_α of an irreducible representation $\alpha \colon \pi_1(M) \to \mathrm{SL}(n,\mathbb{C})$ is fixed under the $\mathbb{Z}/n$ action if and only if α is metabelian.*

Theorem 12 goes a little further in that it gives a characterization of the entire fixed point set $X_n(N_K)^{\mathbb{Z}/n}$ in terms of characters ξ_α of the metabelian representations $\alpha = \alpha_{(n,\chi)}$ described in Subsection 2.3. When $n = 2$, it turns out that every metabelian representation is dihedral and in this case Theorem 1 was first proved by F. Nagasato and Y. Yamaguchi (cf. [NY08, Proposition 4.8]).

Our second result concerns deformations of irreducible metabelian $\mathrm{SL}(n,\mathbb{C})$ representations. In order to formulate it, we need to introduce some additional notation. To any representation $\alpha \colon \pi_1(M) \to \mathrm{SL}(n,\mathbb{C})$ we can associate the twisted homology and cohomology groups with coefficients in the Lie algebra $sl(n,\mathbb{C})$ endowed with the adjoint action of $\pi_1(M)$ given by $g \cdot A := \alpha(g)A\alpha(g)^{-1}$ for $g \in \pi_1(M)$ and $A \in sl(n,\mathbb{C})$. Throughout the paper, we use $sl(n,\mathbb{C})_{ad\,\alpha}$ when referring to $sl(n,\mathbb{C})$ as a $\pi_1(M)$–module via the adjoint action.

In Proposition 15 we show that with respect to the natural symplectic structure on $H^1(\partial N_K; sl(n,\mathbb{C})_{ad\,\alpha})$, the image of $H^1(N_K; sl(n,\mathbb{C})_{ad\,\alpha})$ in $H^1(\partial N_K; sl(n,\mathbb{C})_{ad\,\alpha})$ is Lagrangian. (This is a special case of a general phenomenon elucidated nicely by A. Sikora in [Si09].) Because $\dim_\mathbb{C} H^1(\partial N_K; sl(n,\mathbb{C})_{ad\,\alpha}) = 2n-2$, and since the image of $H^1(N_K; sl(n,\mathbb{C})_{ad\,\alpha})$ must be half–dimensional, we see that $\dim_\mathbb{C} H^1(N_K; sl(n,\mathbb{C})_{ad\,\alpha}) \geq n - 1$.

In the case that equality holds, i.e. if $\dim_\mathbb{C} H^1(N_K; sl(n,\mathbb{C})_{ad\,\alpha}) = n-1$, Lemma 16 says that α has finite image (in particular α is conjugate to a unitary representation) and the following theorem says that α can be deformed within a smooth $(n-1)$–dimensional family of conjugacy classes of irreducible $\mathrm{SL}(n,\mathbb{C})$ representations.

Theorem 2. *Suppose $\alpha \colon \pi_1(N_K) \to \mathrm{SL}(n,\mathbb{C})$ is an irreducible metabelian representation such that $\dim_\mathbb{C} H^1(N_K; sl(n,\mathbb{C})_{ad\,\alpha}) = n - 1$. Then the following hold:*

(i) *The character ξ_α is a smooth point in the character variety $X_n(N_K)$ and there exists a smooth complex $(n-1)$–dimensional family of characters of irreducible $\mathrm{SL}(n,\mathbb{C})$ representations near $\xi_\alpha \in X_n(N_K)$.*

(ii) *If α is a unitary representation, then ξ_α is a smooth point in $X_{SU(n)}(N_K)$ and there exists a smooth real $(n-1)$–dimensional family of characters of irreducible $SU(n)$ representations near $\xi_\alpha \in X_{SU(n)}(N_K)$.*

Finally both of the deformation families can be chosen so that ξ_α is the only metabelian character.

The key step in Theorem 2 is presented in Theorem 17, which is proved by developing an SL(n, $\mathbb{C}$) version of the deformation argument given originally by M. Heusener, J. Porti, and E. Suàrez Peirò in [HPS01] for SL(2, $\mathbb{C}$). The same result was proved using similar methods by L. Abdelghani, M. Heusener, and H. Jebali (cf. [AHJ07, Theorem 8]).

Recall that Lemma 16 also implies α is conjugate to a unitary representation, and once we choose a unitary representative we have that $\dim_{\mathbb{R}} H^1(N_K; su(n)_{ad\,\alpha}) = \dim_{\mathbb{C}} H^1(N_K; sl(n, \mathbb{C})_{ad\,\alpha}) = n - 1$. An application of Theorem 19 now shows that α can be deformed within a real $(n-1)$–dimensional family of conjugacy classes of irreducible $SU(n)$ representations.

It would be interesting to study those metabelian representations α for which Theorem 2 applies. For example, the results of [Na07, Bu90] show that, if K is a 2-bridge knot, then every irreducible metabelian representation $\alpha\colon \pi_1(N_K) \to$ SL(2, $\mathbb{C}$) satisfies $H^1(N_K; sl(2, \mathbb{C})_{ad\,\alpha}) = \mathbb{C}$. One could attempt direct computations of $H^1(N_K; sl(n, \mathbb{C})_{ad\,\alpha})$ for irreducible metabelian representations of other knots and for $n > 2$. We do not take up this problem here, but instead content ourselves with the next result, which provides an alternative criterion for the condition $\dim_{\mathbb{C}} H^1(N_K; sl(n, \mathbb{C})_{ad\,\alpha}) = n - 1$ of Theorem 2 to hold.

Suppose that n is chosen so that $b_1(L_n) = 0$ (e.g. n is a prime power) and let $\alpha\colon \pi_1(N_K) \to$ SL(n, $\mathbb{C}$) be an irreducible metabelian representation. Suppose that $\varphi\colon \pi_1(N_K) \to \mathbb{Z}/n \ltimes H$ is a group homomorphism with H finite and abelian such that α factors through φ. For example we could take φ to be the natural surjection $\pi_1(N_K) \to \mathbb{Z}/n \ltimes H_1(L_n)$. The following theorem has been proved in the case $\mathbb{Z}/n \ltimes H$ is a dihedral group by Boileau and Boyer (cf. [BB07, Lemma A.2]).

Theorem 3. *Suppose that n is such that $b_1(L_n) = 0$. Let $\alpha\colon \pi_1(N_K) \to$ SL(n, $\mathbb{C}$) be an irreducible metabelian representation and $\varphi\colon \pi_1(N_K) \to \mathbb{Z}/n \ltimes H$ be a group homomorphism with H finite and abelian such that α factors through φ. Denote by $\widetilde{N}_\varphi \to N_K$ the covering map corresponding to φ. Then the following hold:*

(i) *$b_1(\widetilde{N}_\varphi) \geq |H|$ and if $b_1(\widetilde{N}_\varphi) = |H|$, then $\dim_{\mathbb{C}} H^1(N_K; sl(n, \mathbb{C})_{ad\,\alpha}) = n - 1$.*
(ii) *The cover $\widetilde{N}_\varphi \to N_K$ extends to a cover $\widehat{\Sigma}_\varphi \to \Sigma$ branched over K.*
(iii) *If $b_1(\widehat{\Sigma}_\varphi) = 0$, then $\dim_{\mathbb{C}} H^1(N_K; sl(n, \mathbb{C})_{ad\,\alpha}) = n - 1$.*

2. The classification of metabelian representations of knot groups

In this section we recall some results from [BF08] regarding the classification of metabelian representations of knot groups (see also [Fr04]), and we provide some general results on the twisted homology and cohomology groups.

2.1. **Preliminaries.** Given a group π, we shall write $\pi^{(n)}$ for the n–th term of the derived series of π. These subgroups are defined inductively by setting $\pi^{(0)} = \pi$ and $\pi^{(i+1)} = [\pi^{(i)}, \pi^{(i)}]$. The group π is called *metabelian* if $\pi^{(2)} = \{e\}$.

Suppose V is a finite dimensional vector space over $\mathbb{C}$. A representation $\varrho\colon \pi \to \mathrm{Aut}(V)$ is called *metabelian* if ϱ factors through $\pi/\pi^{(2)}$. The representation ϱ is called *reducible* if there exists a proper subspace $U \subset V$ invariant under $\varrho(\gamma)$ for all $\gamma \in \pi$. Otherwise ϱ is called *irreducible* or *simple*. If ϱ is the direct sum of simple representations, then ϱ is called *semisimple*.

Two representations $\varrho_1\colon \pi \to \mathrm{Aut}(V)$ and $\varrho_2\colon \pi \to \mathrm{Aut}(W)$ are called *isomorphic* if there exists an isomorphism $\phi\colon V \to W$ such that $\phi^{-1} \circ \varrho_1(g) \circ \phi = \varrho_2(g)$ for all $g \in \pi$.

2.2. **Metabelian quotients of knot groups.** Let $K \subset \Sigma^3$ be a knot in an integral homology 3-sphere. In the following we denote by $\widetilde{N}_K$ the infinite cyclic cover of N_K corresponding to the abelianization $\pi_1(N_K) \to H_1(N_K) \cong \mathbb{Z}$. Therefore $\pi_1(\widetilde{N}_K) = \pi_1(N_K)^{(1)}$ and

$$H_1(N_K; \mathbb{Z}[t^{\pm 1}]) = H_1(\widetilde{N}_K) \cong \pi_1(N_K)^{(1)}/\pi_1(N_K)^{(2)}.$$

The $\mathbb{Z}[t^{\pm 1}]$–module structure is given on the right hand side by $t^n \cdot g := \mu^{-n} g \mu^n$, where μ is a meridian of K.

For a knot K, we set $\pi := \pi_1(N_K)$ and consider the short exact sequence

$$1 \to \pi^{(1)}/\pi^{(2)} \to \pi/\pi^{(2)} \to \pi/\pi^{(1)} \to 1.$$

Since $\pi/\pi^{(1)} = H_1(N_K) \cong \mathbb{Z}$, this sequence splits and we get isomorphisms

$$\begin{array}{ccccc}
\pi/\pi^{(2)} & \cong & \pi/\pi^{(1)} \ltimes \pi^{(1)}/\pi^{(2)} & \cong & \mathbb{Z} \ltimes \pi^{(1)}/\pi^{(2)} \cong \mathbb{Z} \ltimes H_1(N_K; \mathbb{Z}[t^{\pm 1}]) \\
g & \mapsto & (\mu^{\varepsilon(g)}, \mu^{-\varepsilon(g)}g) & \mapsto & (\varepsilon(g), \mu^{-\varepsilon(g)}g),
\end{array}$$

where the semidirect products are taken with respect to the $\mathbb{Z}$ actions defined by letting $n \in \mathbb{Z}$ act by conjugation by μ^n on $\pi^{(1)}/\pi^{(2)}$ and by multiplication by t^n on $H_1(N_K; \mathbb{Z}[t^{\pm 1}])$. This demonstrates the following lemma.

Lemma 4. *Given a knot K, the set of metabelian representations of $\pi_1(N_K)$ can be canonically identified with the set of representations of $\mathbb{Z} \ltimes H_1(N_K; \mathbb{Z}[t^{\pm 1}])$.*

2.3. **Irreducible metabelian** SL($n,\mathbb{C}$) **representations of knot groups.** Let K be a knot. We write $H = H_1(N_K; \mathbb{Z}[t^{\pm 1}])$. In order to classify the irreducible metabelian SL($n,\mathbb{C}$) representations of $\pi_1(N_K)$ it suffices by Lemma 4 to classify the irreducible SL($n,\mathbb{C}$) representations of $\mathbb{Z} \ltimes H$.

Throughout this paper we will frequently make use of the following well–known facts:

(1) H is a finitely generated $\mathbb{Z}[t^{\pm 1}]$–module such that multiplication by $t - 1$ is an isomorphism.

(2) Given $n \in \mathbb{N}$ there exists a natural identification $H/(t^n - 1) \cong H_1(L_n)$, where L_n denotes the n–fold cover of Σ^3 branched over K.

Let $\chi \colon H \to \mathbb{C}^*$ be a character which factors through $H/(t^n - 1)$ and $z \in U(1)$. Then it follows from [BF08, Section 3] that, for $(j, h) \in \mathbb{Z} \ltimes H$, setting

$$
\alpha_{(n,\chi,z)}(j,h) = \begin{pmatrix} 0 & & \ldots & z \\ z & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & z & 0 \end{pmatrix}^j \begin{pmatrix} \chi(h) & 0 & \ldots & 0 \\ 0 & \chi(th) & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & \chi(t^{n-1}h) \end{pmatrix}
$$

defines a GL($n,\mathbb{C}$) representation. Note that $\alpha_{(n,\chi,z)}$ factors through $\mathbb{Z} \ltimes H/(t^n - 1)$. We are mostly interested in the special values for z:

(1) If $z \in \mathbb{C}$ such that $z^n = (-1)^{n+1}$, then we write $\alpha_{(n,\chi)} = \alpha_{(n,\chi,z)}$. Note that $\alpha_{(n,\chi)}$ defines an SL($n,\mathbb{C}$)–representation and that the isomorphism type of this representation is independent of the choice of z.

(2) If $z = 1$, then we write $\beta_{(n,\chi)} = \alpha_{(n,\chi,1)}$.

In our notation we will normally not distinguish between metabelian representations of $\pi_1(N_K)$ and representations of $\mathbb{Z} \ltimes H$.

In the following we say that a character $\chi \colon H \to \mathbb{C}^*$ has *order* n if it factors through $H/(t^n - 1)$ and but not through $H/(t^\ell - 1)$ for any $\ell < n$. Given a character $\chi \colon H \to \mathbb{C}^*$, let $t^i \chi$ be the character defined by $(t^i \chi)(h) = \chi(t^i h)$. Any character $\chi \colon H \to \mathbb{C}^*$ which factors through $H/(t^n - 1)$ must have order k for some divisor k of n. The following is a combination of [BF08, Lemma 2.2] and [BF08, Theorem 3.3].

Theorem 5. *Suppose $\chi \colon H \to \mathbb{C}^*$ is a character that factors through $H/(t^n - 1)$.*

(i) *$\alpha_{(n,\chi)} \colon \mathbb{Z} \ltimes H \to$ SL($n,\mathbb{C}$) is irreducible if and only if the character χ has order n.*

(ii) *If $\chi \colon H \to \mathbb{C}^*$ is a character of order n, then $\alpha_{(n,\chi)}$ defines an irreducible SL($n,\mathbb{C}$) representation.*

(iii) *Given two characters $\chi, \chi' \colon H \to \mathbb{C}^*$ of order n, the representations $\alpha_{(n,\chi)}$ and $\alpha_{(n,\chi')}$ are conjugate if and only if $\chi = t^k \chi'$ for some k.*

(iv) *For any irreducible representation $\alpha \colon \mathbb{Z} \ltimes H \to$ SL($n,\mathbb{C}$) there exists a character $\chi \colon H \to \mathbb{C}^*$ of order n such that α is conjugate to $\alpha_{(n,\chi)}$.*

2.4. **The adjoint representation corresponding to a metabelian representation.** The following lemma allows us to decompose the adjoint representation corresponding to a metabelian representation into a direct sum of much simpler representations.

Lemma 6. *Let K be a knot, $n \in \mathbb{N}$ and $\chi \colon H_1(L_n) \to \mathbb{C}^*$ be a character. Suppose $\alpha = \alpha_{(n,\chi)}$ and let $\theta_1 \colon \pi_1(N_K) \to \mathrm{GL}(1, \mathbb{C})$ be the trivial representation. Setting $\alpha_n \colon \pi_1(N_K) \to \mathrm{Aut}(\mathbb{C}[\mathbb{Z}/n])$ equal to the regular representation corresponding to the canonical projection map $\pi_1(N_K) \to \mathbb{Z} \to \mathbb{Z}/n$ and using $\mathrm{ad}\,\alpha \colon \pi_1(N_K) \to \mathrm{Aut}(sl(n, \mathbb{C}))$ to denote the adjoint representation, we have the following isomorphism of representations:*

$$\mathrm{ad}\,\alpha \oplus \theta_1 \cong \alpha_n \oplus \bigoplus_{i=1}^{n-1} \beta_{(n,\chi_i)},$$

where χ_i is the character defined by $\chi_i(v) := \chi(v)^{-1}\chi(t^i v)$. Furthermore, if χ is a character of order n, then $\chi_1, \ldots, \chi_{n-1}$ are also characters of order n.

Proof. As before, we write $\pi = \pi_1(N_K)$. We denote by $\beta \colon \pi \to \mathrm{Aut}(gl(n, \mathbb{C}))$ the adjoint representation of α on $gl(n, \mathbb{C})$, so $\beta(g)(A) = \alpha(g)A\alpha(g)^{-1}$ for $g \in \pi$ and $A \in gl(n, \mathbb{C})$. Note that $gl(n, \mathbb{C}) = sl(n, \mathbb{C}) \oplus \mathbb{C} \cdot I$. It follows immediately that $\beta = \mathrm{ad}\,\alpha \oplus \theta_1$ splits off a trivial factor. It therefore suffices to show that

$$\beta \cong \alpha_n \oplus \bigoplus_{i=1}^{n-1} \beta_{(n,\chi_i)}.$$

For $i = 0, \ldots, n-1$ we denote by V_i the set of all matrices (a_{jk}) with $a_{jk} = 0$ unless $j - k \equiv i\,(n)$. It is not difficult to see that the action of π on $gl(n, \mathbb{C})$ restricts to actions on $V_0, V_1, \ldots, V_{n-1}$. We equip V_i with the ordered basis $\{e_{i+1,1}, e_{i+2,2}, \ldots, e_{i+n,n}\}$, where the indices are taken modulo n. The restriction of β to V_i can then be calculated with respect to this basis and it is straightforward to verify that it is given by $\beta_{(n,\chi_i)}$. Note that the character χ_0 is the trivial character, and therefore $\beta_{(n,\chi_0)} = \alpha_n$. $\square$

3. Twisted homology and cohomology groups

3.1. **Definitions and basic properties.** We recall the definition of twisted homology and cohomology and their basic properties. Let (X, Y) be a pair of topological spaces, V a finite dimensional complex vector space and $\alpha \colon \pi_1(X) \to \mathrm{Aut}(V)$ a representation. Denote by $p \colon \widetilde{X} \to X$ the universal covering and set $\widetilde{Y} := p^{-1}(Y)$. Letting $\pi = \pi_1(X)$, we use the representation α to regard V as a left $\mathbb{Z}[\pi]$–module. The chain complex $C_*(\widetilde{X}, \widetilde{Y})$ is also a left $\mathbb{Z}[\pi]$–module via deck transformations and we can form the twisted cohomology groups

$$H^*(X, Y; V_\alpha) = H_*(\mathrm{Hom}_{\mathbb{Z}[\pi]}(C_*(\widetilde{X}, \widetilde{Y}), V)).$$

We write $b^i(X, Y; V_\alpha) = \dim_{\mathbb{C}} H^i(X, Y; V_\alpha)$. Using the natural involution $g \mapsto g^{-1}$ on the group ring $\mathbb{Z}[\pi]$, we can also view $C_*(\widetilde{X}, \widetilde{Y})$ as a right $\mathbb{Z}[\pi]$–module and form the twisted homology groups

$$H_*(X, Y; V_\alpha) = H_*(C_*(\widetilde{X}, \widetilde{Y}) \otimes_{\mathbb{Z}[\pi]} V).$$

We write $b_i(X, Y; V_\alpha) = \dim_{\mathbb{C}} H_i(X, Y; V_\alpha)$.

Notice first off that the twisted cohomology and homology groups corresponding to isomorphic representations are isomorphic as $\mathbb{C}$–vector spaces. Note further that the 0–th twisted groups of a space can be computed immediately from the fundamental group (cf. [HS97, Section VI]):

(1)
$$
\begin{aligned}
H^0(X; V_\alpha) &= \{v \in V \,|\, \alpha(g)v = v \text{ for all } g \in \pi_1(X)\}, \\
H_0(X; V_\alpha) &= V/\{\alpha(g)v - v \,|\, v \in V, g \in \pi_1(X)\}.
\end{aligned}
$$

Finally note that if M is an n–manifold, then by Poincaré duality we have

$$H_i(M; V_\alpha) \cong H^{n-i}(M, \partial M; V_\alpha) \text{ and } H_i(M, \partial M; V_\alpha) \cong H^{n-i}(M; V_\alpha).$$

We will several times make use of the following well–known lemma (cf. e.g. [FK06, Lemma 2.3]).

Lemma 7. *Suppose that V is equipped either with a bilinear non–singular form or with a hermitian non–singular form, and that α is orthogonal respectively unitary with respect to this form. Then*

$$H_i(X, Y; V_\alpha) \cong H^i(X, Y; V_\alpha)$$

for any i.

The next lemma shows that the hypotheses of the previous lemma is satisfied in the case of the adjoint representation.

Lemma 8. *Let $\alpha\colon \pi \to \mathrm{SL}(n, \mathbb{C})$ be a representation and consider the corresponding adjoint representation $\mathrm{ad}\,\alpha\colon \pi \to \mathrm{Aut}(sl(n, \mathbb{C}))$. Then there exists a non–singular bilinear form on $sl(n, \mathbb{C})$ with respect to which $\mathrm{ad}\,\alpha$ is orthogonal.*

Proof. It is straightforward to verify that the form $(A, B) \mapsto tr(A \cdot B)$ has all the required properties. $\square$

3.2. **Calculations.** In this subsection we present some calculations of twisted homology and cohomology groups that will be used in the proofs of the main results.

Lemma 9. *Let K be a knot, $\chi\colon H_1(L_n) \to \mathbb{C}^*$ a character and $z \in U(1)$. Let $V = \mathbb{C}^n$ and $\alpha = \alpha_{(n, \chi, z)}\colon \pi_1(N_K) \to \mathrm{Aut}(V)$, and set $\hat{\alpha}$ to be the restriction of α to $\pi_1(\partial N_K)$. If $z^n = 1$, then the following hold:*

$$
\begin{aligned}
b^0(\partial N_K; V_{\hat{\alpha}}) &= b_0(\partial N_K; V_{\hat{\alpha}}) &= 1, \\
b^1(\partial N_K; V_{\hat{\alpha}}) &= b_1(\partial N_K; V_{\hat{\alpha}}) &= 2, \\
b^2(\partial N_K; V_{\hat{\alpha}}) &= b_2(\partial N_K; V_{\hat{\alpha}}) &= 1.
\end{aligned}
$$

Proof. We let μ and λ be the meridian and longitude of K. Note that $\alpha(\lambda)$ is trivial and that $\alpha(\mu)$ is diagonal with eigenvalues $z, ze^{2\pi i/n}, \ldots, ze^{2\pi i(n-1)/n}$, which are distinct. Note that $\alpha(\mu)$ has precisely one eigenvalue which equals one. A direct calculation using Equation (1) now shows that $b^0(\partial N_K, V_{\hat{a}}) = 1$ and $b_0(\partial N_K, V_{\hat{a}}) = 1$. It follows from duality that $b_2(\partial N_K; V_{\hat{a}}) = 1$ and $b^2(\partial N_K; V_{\hat{a}}) = 1$. Since the Euler characteristic of the torus ∂N_K is zero it now follows that $b^1(\partial N_K; V_{\hat{a}}) = b_1(\partial N_K; V_{\hat{a}}) = 2$. $\qquad\square$

Lemma 10. *Let K be a knot. For $i = 1, \ldots, \ell$, let $\chi_i \colon H_1(L_n) \to \mathbb{C}^*$ be a non–trivial character and $z_i \in U(1)$ with $z_i^n = 1$. Let $V = \mathbb{C}^{n\ell}$ and consider the representation $\alpha = \bigoplus_{i=1}^{\ell} \alpha_{(n,\chi_i,z_i)} \colon \pi_1(N_K) \to \mathrm{Aut}(V)$. Then the following hold:*

(i) $b^0(N_K; V_\alpha) = 0$,

(ii) *if α is orthogonal or unitary with respect to a non–singular form on V, then* $b^1(N_K; V_\alpha) \geq \ell$.

Proof. The first statement is an immediate consequence of Equation (1) an the assumption that χ_i are non-trivial. By Lemma 9 we have $b_1(\partial N_K; V_\alpha) = 2\ell$. Now consider the following short exact sequence

$$H^1(N_K; V_\alpha) \to H^1(\partial N_K; V_\alpha) \to H^2(N_K, \partial N_K; V_\alpha).$$

It follows that either $b^1(N_K; V_\alpha) \geq \ell$ or $b^2(N_K, \partial N_K; V_\alpha) \geq \ell$. But by Poincaré duality and by Lemma 7 the latter also equals $b^1(N_K; V_\alpha)$. $\qquad\square$

4. PROOFS OF MAIN RESULTS

4.1. **Proof of Theorem 1.** Set $\omega = e^{2\pi i/n}$ and recall the action of the cyclic group $\mathbb{Z}/n = \langle \sigma \mid \sigma^n = 1 \rangle$ on representations $\alpha \colon \pi_1(N_K) \to \mathrm{SL}(n, \mathbb{C})$ obtained by setting $(\sigma \cdot \alpha)(g) = \omega^{\varepsilon(g)} \alpha(g)$ for all $g \in \pi_1(N_K)$, where $\varepsilon \colon \pi_1(N_K) \to H_1(N_K) = \mathbb{Z}$.

We begin with the following useful lemma.

Lemma 11. *Suppose $\alpha \colon \pi_1(N_K) \to \mathrm{SL}(n, \mathbb{C})$ is a representation whose associated character $\xi_\alpha \in X_n(N_K)$ is a fixed point of the $\mathbb{Z}/n$ action. Then up to conjugation, we have*

$$(2) \qquad \alpha(\mu) = \begin{pmatrix} 0 & & \cdots & z \\ z & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & z & 0 \end{pmatrix},$$

for some (in fact any) $z \in U(1)$ such that $z^n = (-1)^{n+1}$.

Proof. Let $c(t) = \det(\alpha(\mu) - tI)$ denote the characteristic polynomial of $\alpha(\mu)$, which we can write as

$$c(t) = (-1)^n t^n + c_{n-1} t^{n-1} + \cdots + c_1 t + 1.$$

Note that $c(t)$ is determined by the character $\xi_\alpha \in X_n(N_K)$, and so assuming ξ_α is a fixed point of the $\mathbb{Z}/n$ action, we conclude that $\alpha(\mu)$ and $\omega^k\alpha(\mu)$ have the same characteristic polynomials for all k. In particular,

$$
\begin{aligned}
c(t) &= \det(\omega^{-1}\alpha(\mu) - tI) \\
&= \det(\omega^{-1}\alpha(\mu) - (\omega^{-1}\omega)tI) \\
&= \det(\omega^{-1}I)\det(\alpha(\mu) - \omega tI) \\
&= \det(\alpha(\mu) - t\omega I) = c(\omega t).
\end{aligned}
$$

However, $\omega^k \neq 1$ unless $n|k$, and this implies $0 = c_{n-1} = c_{n-2} = \cdots = c_1$ and $c(t) = (-1)^n t^n + 1$. In particular the matrix $\alpha(\mu)$ and the matrix appearing in Equation (2) have the same set of n distinct eigenvalues. This implies that the two matrices are conjugate. $\qquad\square$

In order to prove Theorem 1, we shall establish the following slightly more general result.

Theorem 12. *The fixed point set of the $\mathbb{Z}/n$ action on $X_n(N_K)$ consists of characters ξ_α of the metabelian representations $\alpha = \alpha_{(n,\chi)}$ described in Theorem 5. In other words,*

$$
X_n(N_K)^{\mathbb{Z}/n} = \{\xi_\alpha \mid \alpha = \alpha_{(n,\chi)} \text{ for } \chi \colon H_1(N_K; \mathbb{Z}[t^{\pm 1}]) \to \mathbb{C}^*\}.
$$

Notice that Theorem 1 can be viewed as the special case of Theorem 12 where $\alpha_{(n,\chi)}$ is irreducible. Notice further that not every reducible metabelian representation is of the form $\alpha_{(n,\chi)}$.

Proof. We first show that if $\alpha \colon \pi_1(N_K) \to \mathrm{SL}(n, \mathbb{C})$ is given as $\alpha = \alpha_{(n,\chi)}$, then $\sigma \cdot \alpha$ is conjugate to α. This of course implies that $\xi_\alpha = \xi_{\sigma \cdot \alpha}$.

Assume then that $\alpha = \alpha_{(n,\chi)}$. Then we have

$$
\alpha(\mu) = \begin{pmatrix} 0 & \cdots & & z \\ z & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & z & 0 \end{pmatrix},
$$

where z satisfies $z^n = (-1)^{n+1}$. Further, $\alpha(g)$ is diagonal for all $g \in [\pi_1(N_K), \pi_1(N_K)]$. By definition of $\sigma \cdot \alpha$, we see that

$$
(\sigma \cdot \alpha)(\mu) = \omega\alpha(\mu) = \begin{pmatrix} 0 & \cdots & & \omega z \\ \omega z & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \omega z & 0 \end{pmatrix}
$$

and that $(\sigma \cdot \alpha)(g) = \alpha(g)$ for all $g \in [\pi_1(N_K), \pi_1(N_K)]$. It follows easily from Theorem 5 that $\sigma \cdot \alpha$ and $\alpha_{(n,\chi)}$ are conjugate; however it is easy to see this directly too. Simply

take

$$P = \begin{pmatrix} 1 & & & 0 \\ & \omega & & \\ & & \ddots & \\ 0 & & & \omega^{n-1} \end{pmatrix},$$

and compute that $\sigma \cdot \alpha = P\alpha P^{-1}$ as claimed.

We now show the other implication, namely that each point $\xi \in X_n(N_K)^{\mathbb{Z}/n}$ in the fixed point set can be represented as the character $\xi = \xi_\alpha$ of a metabelian representation $\alpha = \alpha_{(n,\chi)}$. Here $\chi\colon H_1(N_K; \mathbb{Z}[t^{\pm 1}]) \to \mathbb{C}^*$ is a character that factors through $H_1(N_K; \mathbb{Z}[t^{\pm 1}])/(t^n - 1)$, so it has order k for some k dividing n. Recall that Theorem 5 tells us that $\alpha_{(n,\chi)}$ is irreducible if and only if χ has order n.

By the general results on representation spaces and character varieties (see [LM85]), it follows that every point in the character variety $X_n(N_K)$ can be represented as ξ_α for some semisimple representation $\alpha\colon \pi_1(N_K) \to \mathrm{SL}(n, \mathbb{C})$. Further, we see that two semisimple representations α_1 and α_2 determine the same character if and only if α_1 is conjugate to α_2. (This is evident from the fact that the orbits of the semisimple representations under conjugation are closed.)

Given $\xi \in X_n(N_K)^{\mathbb{Z}/n}$, we can therefore suppose that $\xi = \xi_\alpha$ for some semisimple representation α. Clearly $\sigma \cdot \alpha$ is also semisimple, and since $\xi_\alpha = \xi_{\sigma \cdot \alpha}$, we conclude that α and $\sigma \cdot \alpha$ are conjugate representations.

Lemma 11 implies $\alpha(\mu)$ is conjugate to the matrix in Equation (2). It is convenient to conjugate α so that $\alpha(\mu)$ is diagonal, meaning that

$$\alpha(\mu) = \begin{pmatrix} z & & & 0 \\ & \omega z & & \\ & & \ddots & \\ 0 & & & \omega^{n-1} z \end{pmatrix},$$

where z satisfies $z^n = (-1)^{n+1}$.

Since the conjugacy class of α is fixed under the $\mathbb{Z}/n$ action, it follows that there exists a matrix $A \in \mathrm{SL}(n, \mathbb{C})$ such that $A\alpha A^{-1} = \sigma \cdot \alpha$. In other words, for all $g \in \pi_1(N_K)$, we have

$$A\alpha(g)A^{-1} = \omega^{\varepsilon(g)}\alpha(g).$$

In the case of the meridian, this shows

$$A\alpha(\mu) = \omega\alpha(\mu)A,$$

which implies $A = (a_{ij})$ satisfies $a_{ij} = 0$ unless $j = i+1 \mod (n)$. Thus, we see that

$$A = \begin{pmatrix} 0 & \lambda_1 & 0 & \dots & 0 \\ 0 & 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \lambda_{n-1} \\ \lambda_n & 0 & \dots & & 0 \end{pmatrix}$$

for some $\lambda_1, \dots, \lambda_n$ satisfying $\lambda_1 \cdots \lambda_n = (-1)^{n+1}$.

It is completely straightforward to see that the characteristic polynomial of A is given by

$$\det(A - tI) = (-1)^n (t^n - (-1)^{n+1}).$$

From this, we conclude that A has as its eigenvalues the n distinct n–th roots of $(-1)^{n+1}$. In particular, the subset of SL($n,\mathbb{C}$) of matrices that commute with A is just a copy of the unique maximal torus $T_A \cong (\mathbb{C}^*)^{n-1}$ containing A.

For any $g \in [\pi_1(N_K), \pi_1(N_K)]$, we have $\alpha(g) = (\sigma \cdot \alpha)(g)$. Thus it follows that $A\alpha(g)A^{-1} = \alpha(g)$, and this implies that $\alpha(g) \in T_A$ for all $g \in [\pi_1(N_K), \pi_1(N_K)]$. This shows that the restriction of α to the commutator subgroup $[\pi_1(N_K), \pi_1(N_K)]$ is abelian, and we conclude from this that α is indeed metabelian. Notice that this, and an application of Theorem 5, completes the proof in the case α is irreducible.

In the general case, we see by Lemma 4 that α factors through $\mathbb{Z} \ltimes H_1(N_K; \mathbb{Z}[t^{\pm 1}])$. Let $H = H_1(N_K; \mathbb{Z}[t^{\pm 1}])$. Given a character $\chi \colon H \to \mathbb{C}^*$ we define the associated weight space V_χ by setting

$$V_\chi = \{v \in \mathbb{C}^n \mid \chi(h) \cdot v = \alpha(h)v \text{ for all } h \in H\}.$$

Recall that $A \cdot \alpha(h) \cdot A^{-1} = \alpha(h)$ for any $h \in H$. It is straightforward so show that A restricts to an automorphism of V_χ. Since H is abelian there exists at least one character $\chi \colon H \to \mathbb{C}^*$ such that V_χ is non–trivial. Given i we denote by $t^i \chi$ the character given by $(t^i \chi)(h) = \chi(t^i h), h \in H$.

Note that A has n distinct eigenvalues and therefore is diagonalizable. Since A restricts to an automorphism of V_χ, there is an eigenvector v of A which lies in V_χ. Let λ be the corresponding eigenvalue. By the proof of [BF08, Theorem 2.3], the map $\alpha(\mu)$ induces an isomorphism $V_\chi \to V_{t\chi}$. We now calculate

$$A \cdot \alpha(\mu)v = (A\alpha(\mu)A^{-1}) \cdot Av = \omega\alpha(\mu) \cdot \lambda v = \lambda\omega \cdot \alpha(\mu)v,$$

i.e. $\alpha(\mu)v \in V_{t\chi}$ is an eigenvector of A with eigenvalue $\omega\lambda$.

Iterating this argument, we see that $\alpha(\mu)^i v$ lies in $V_{t^i \chi}$ and is an eigenvector of A with eigenvalue $\omega^i \lambda$. Since ω is a primitive n–th root of unity, the eigenvalues $\lambda, \omega\lambda, \dots, \omega^{n-1}\lambda$ are all distinct, and this implies that the corresponding eigenvectors $v, \alpha(\mu)v, \dots, \alpha(\mu)^{n-1}v$ form a basis for $\mathbb{C}^n$.

Let m be the order of χ, i.e. m is the minimal number such that $\chi = t^m \chi$. By the above we have that $\mathbb{C}^n$ is generated by $V_\chi, V_{t\chi}, \dots, V_{t^m \chi}$. Since the characters

$\chi, t\chi, \ldots, t^m\chi$ are pairwise distinct, it follows that $\mathbb{C}^n$ is given as the direct sum $V_\chi \oplus V_{t\chi} \oplus \cdots \oplus V_{t^{m-1}\chi}$.

We write $k = \dim_{\mathbb{C}}(V_\chi)$ and note that $n = km$. We note further that $\alpha(\mu)^m$ has eigenvalues given by the set

$$(3) \qquad \{z^m, z^m e^{2\pi i/k}, \ldots, z^m e^{2\pi i(k-1)/k}\},$$

and each eigenvalue has multiplicity m. Clearly $\alpha(\mu)^m$ restricts to an automorphism of $V_{t^i\chi}$ for $i = 0, \ldots, m-1$, and equally clearly we see that the restrictions all give conjugate representations. This implies that the restriction of $\alpha(\mu)^m$ to V_χ has eigenvalues in the set (3) above, each occurring with multiplicity 1. In particular we can find a basis $\{v_1, \ldots, v_k\}$ for V_χ in which the matrix of $\alpha(\mu)^m$ has the form

$$\alpha(\mu^m) = \begin{pmatrix} 0 & \ldots & & z^m \\ z^m & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & z^m & 0 \end{pmatrix}.$$

It is now straightforward to verify that with respect to the ordered basis

$$\left\{ \begin{array}{cccc} v_1, & z^{-1}\alpha(\mu)v_1, & \ldots, & z^{-(m-1)}\alpha(\mu)^{m-1}v_1, \\ v_2, & z^{-1}\alpha(\mu)v_2, & \ldots, & z^{-(m-1)}\alpha(\mu)^{m-1}v_2, \\ \vdots & \vdots & \ldots & \vdots \\ v_k, & z^{-1}\alpha(\mu)w_k, & \ldots, & z^{-(m-1)}\alpha(\mu)^{m-1}v_k \end{array} \right\},$$

α is given by $\alpha(n, \chi)$. $\qquad\square$

We pause briefly to explain how Theorem 12 can be applied to see that the twisted Alexander polynomials at metabelian representations must be a polynomial in t^n.

In the following, we use $\Delta^\alpha_{K,i}(t)$ to denote the i–th twisted Alexander polynomial for a given representation $\alpha \colon \pi_1(N_K) \to \mathrm{SL}(n, \mathbb{C})$ as presented in [FV09].

Proposition 13. *Let α be a metabelian representation of the form $\alpha = \alpha_{n,\chi} \colon \pi_1(N_K) \to \mathrm{SL}(n, \mathbb{C})$. Then*

$$\Delta^\alpha_{K,0}(t) = \begin{cases} 1 - t^n, & \text{if } \chi \text{ is trivial,} \\ 1, & \text{otherwise.} \end{cases}$$

Furthermore the twisted Alexander polynomial $\Delta^\alpha_{K,1}(t)$ is actually a polynomial in t^n.

Remark 14. This result addresses some questions raised by M. Hirasawa and K. Murasugi in their computations of the twisted Alexander polynomials in [HM09]. In particular, it gives a positive answer to Conjecture A from [HM09]. There are other approaches to prove Proposition 13; for example on p. 10 of [HKL08], C. Herald, P. Kirk, and C. Livingston prove the second statement using different techniques.

Proof. The first statement is obvious in the case that χ is trivial. If χ is non–trivial then the statement follows either from a direct calculation.

We now turn to the proof of the second statement. For $\theta \in U(1)$ and any representation $\beta : \pi_1(N_K) \to \mathrm{GL}(n, \mathbb{C})$, define the θ-twist of β to be the representation sending $g \in \pi_1(N_K)$ to $\theta^{\varepsilon(g)}\beta(g)$, where $\varepsilon \colon \pi_1(N_K) \to \mathbb{Z}$ is determined by the orientation of K. We denote the newly obtained representation by $\beta_\theta \colon \pi_1(N_K) \to \mathrm{GL}(n, \mathbb{C})$. Note that in case $\alpha : \pi_1(N_K) \to \mathrm{SL}(n, \mathbb{C})$ and $\theta = e^{2\pi i k/n}$ is an n-th root of unity, α_θ is again an $\mathrm{SL}(n, \mathbb{C})$ representation. The proof of the proposition relies on the formula

$$(4) \qquad\qquad \Delta_{K,1}^{\beta_\theta}(t) = \Delta_{K,1}^{\beta}(\theta t).$$

This formula is well-known and follows directly from the definition of the twisted Alexander polynomial. Equation (4) combines with Theorem 1 to complete the proof, as we now explain. Take $\omega = e^{2\pi i/n}$. If $\alpha = \alpha_{(n,\chi)}$ is metabelian, then Theorem 1 shows that its conjugacy class is fixed under the $\mathbb{Z}/n$ action. In particular, since α and α_ω are conjugate, Equation (4) shows that

$$\Delta_{K,1}^{\alpha}(t) = \Delta_{K,1}^{\alpha_\omega}(t) = \Delta_{K,1}^{\alpha}(\omega t).$$

Expanding $\Delta_{K,1}^{\alpha}(t) = \sum a_i t^i$ and using the fact that $t^k = (\omega t)^k$ if and only if k is a multiple of n, this shows that $a_k = 0$ unless k is a multiple of n and this completes the proof. $\qquad\square$

4.2. **Proof of Theorem 2.** Assume now that $\alpha \colon \pi_1(N_K) \to \mathrm{SL}(n, \mathbb{C})$ is a representation and let $\hat{\alpha} \colon \pi_1(\partial N_K) \to \mathrm{SL}(n, \mathbb{C})$ denote the restriction of α to the boundary torus. Throughout this subsection, we will assume that the image of $\hat{\alpha}$ contains a matrix with n distinct eigenvalues. Note that every irreducible metabelian representation of $\pi_1(N_K)$ satisfies this condition.

Choose $g \in \pi_1(\partial N_K)$ so that $\alpha(g)$ has n distinct eigenvalues. Then this matrix is diagonalizable, and any other matrix that commutes with it must lie in the same maximal torus. Since $\pi_1(\partial N_K) \cong \mathbb{Z} \oplus \mathbb{Z}$ is abelian, we see that the stabilizer subgroup of $\hat{\alpha}$ under conjugation action is again this maximal torus. From this, Poincaré duality and Euler characteristic considerations, we conclude that

$$\begin{aligned}
\dim_{\mathbb{C}} H^0(\partial N_K; sl(n, \mathbb{C})_{ad\,\hat{\alpha}}) &= n - 1, \\
\dim_{\mathbb{C}} H^1(\partial N_K; sl(n, \mathbb{C})_{ad\,\hat{\alpha}}) &= 2(n - 1), \text{ and} \\
\dim_{\mathbb{C}} H^2(\partial N_K; sl(n, \mathbb{C})_{ad\,\hat{\alpha}}) &= n - 1.
\end{aligned}$$

We now consider the long exact sequence in twisted cohomology associated with the pair $(N_K, \partial N_K)$. The inclusions

$$(\partial N_K, \varnothing) \overset{i}{\hookrightarrow} (N_K, \varnothing) \overset{j}{\hookrightarrow} (N_K, \partial N_K)$$

induce the following long exact sequence (coefficients in $sl(n, \mathbb{C})$ twisted by α or $\hat{\alpha}$ understood).

$$0 \longrightarrow H^0(N_K) \longrightarrow H^0(\partial N_K) \longrightarrow H^1(N_K, \partial N_K)$$

(5)
$$\xrightarrow{j^1} H^1(N_K) \xrightarrow{i^1} H^1(\partial N_K) \longrightarrow H^2(N_K, \partial N_K)$$

$$\xrightarrow{j^2} H^2(N_K) \xrightarrow{i^2} H^2(\partial N_K) \to H^3(N_K, \partial N_K) \longrightarrow 0.$$

Exactness of the middle row implies that (with twisted coefficients understood)

$$\dim_{\mathbb{C}} H^1(N_K) + \dim_{\mathbb{C}} H^2(N_K, \partial N_K) \geq \dim_{\mathbb{C}} H^1(\partial N_K) = 2n - 2,$$

and by Poincaré duality and Lemmas 7, 8 we have $\dim_{\mathbb{C}} H^1(N_K) = \dim_{\mathbb{C}} H^2(N_K, \partial N_K)$. This implies $\dim_{\mathbb{C}} H^1(N_K) \geq n - 1$.

The next proposition shows that the image $\left(H^1(N_K) \xrightarrow{i^1} H^1(\partial N_K) \right)$ has dimension $n - 1$, and we explain this by relating it to the general result that for a 3-manifold N with connected boundary ∂N, there is a symplectic structure on the character variety $X_n(\partial N)$ and the image $X_n(N) \to X_n(\partial N)$ is Lagrangian, which is proved for representations into arbitrary reductive groups in the recent paper [Si09] of A. Sikora, under the assumption that ∂X is a connected surface of genus $g \geq 2$. We state and prove the analogous result only for the case of interest to us, namely $\mathrm{SL}(n, \mathbb{C})$ representations of knot complements N_K, but we remark that the argument can be generalized to other Lie groups G (e.g. $G = SU(n)$) and other 3-manifolds N with boundary ∂N a torus.

Proposition 15. *If K is a knot and $\alpha \colon \pi_1(N_K) \to \mathrm{SL}(n, \mathbb{C})$ is a representation such that the image of the restriction $\hat{\alpha} \colon \pi_1(\partial N_K) \to \mathrm{SL}(n, \mathbb{C})$ contains a matrix with n distinct eigenvalues, then the image*

$$\mathrm{image}\left(H^1(N_K; sl(n, \mathbb{C})_{ad\,\alpha}) \xrightarrow{i^1} H^1(\partial N_K; sl(n, \mathbb{C})_{ad\,\hat{\alpha}}) \right)$$

has dimension $n - 1$ and is Lagrangian with respect to the symplectic structure Ω defined below. It follows that

$$\dim_{\mathbb{C}} H^1(N_K; sl(n, \mathbb{C})_{ad\,\alpha}) \geq n - 1.$$

Proof. The fact that $\dim_{\mathbb{C}} \mathrm{image}\left(H^1(N_K) \xrightarrow{i^1} H^1(\partial N_K) \right) = n - 1$ follows easily from a diagram chase of the long exact sequence (5), using the fact that since the image of $\hat{\alpha}$ contains an element with n distinct eigenvalues, we know $H^0(\partial N_K; sl(n, \mathbb{C})_{ad\,\hat{\alpha}}) = \mathbb{C}^{n-1} = H^2(\partial N_K; sl(n, \mathbb{C})_{ad\,\hat{\alpha}})$ and $H^1(\partial N_K; sl(n, \mathbb{C})_{ad\,\hat{\alpha}}) = \mathbb{C}^{2n-2}$.

The symplectic structure Ω on $H^1(\partial N_K; sl(n, \mathbb{C}))$ is defined by composing the cup product with the symmetric bilinear pairing obtained by first multiplying the matrices and then taking the trace:

$$sl(n, \mathbb{C}) \times sl(n, \mathbb{C}) \to gl(n, \mathbb{C}) \to \mathbb{C}$$
$$(A, B) \mapsto A \cdot B \mapsto tr(A \cdot B).$$

We have already seen that the image $\left(H^1(N_K) \xrightarrow{i^1} H^1(\partial N_K) \right)$ has dimension $n-1$, so we just need to show that it is isotropic with respect to Ω.

Suppose $x, y \in H^1(N_K; sl(n, \mathbb{C})_{ad\,\alpha})$ and consider the long exact sequence (5) with untwisted coefficients in $\mathbb{C}$. Let $\overset{\circ}{\cup}$ denote the combined cup and matrix product, so $x \overset{\circ}{\cup} y \in H^2(N_K; gl(n, \mathbb{C})_{ad\,\alpha})$. Using the commutative diagram

$$
\begin{array}{ccc}
H^1(N_K; sl(n, \mathbb{C})_{ad\,\alpha}) \times H^1(N_K; sl(n, \mathbb{C})_{ad\,\alpha}) & \xrightarrow{\overset{\circ}{\cup}} & H^2(N_K; gl(n, \mathbb{C})_{ad\,\alpha}) \\
\downarrow{\scriptstyle i^1 \times i^1} & & \downarrow{\scriptstyle i^2} \\
H^1(\partial N_K; sl(n, \mathbb{C})_{ad\,\hat{\alpha}}) \times H^1(\partial N_K; sl(n, \mathbb{C})_{ad\,\hat{\alpha}}) & \xrightarrow{\overset{\circ}{\cup}} & H^2(\partial N_K; gl(n, \mathbb{C})_{ad\,\hat{\alpha}}),
\end{array}
$$

we see that $\Omega(i^1(x), i^1(y)) = tr(i^1(x) \overset{\circ}{\cup} i^1(y)) = tr\, i^2(x \overset{\circ}{\cup} y)$, so it is in the image of

(6) $$ H^2(N_K; \mathbb{C}) \longrightarrow H^2(\partial N_K; \mathbb{C}), $$

which by exactness of the third row of the long exact sequence (5), now taken with untwisted $\mathbb{C}$ coefficients, equals the kernel of the surjection $H^2(\partial N_K; \mathbb{C}) \to H^3(N_K, \partial N_K; \mathbb{C})$. However, it is not difficult to compute $H^3(N_K, \partial N_K; \mathbb{C}) = \mathbb{C} = H^2(\partial N_K; \mathbb{C})$, and this implies the map in Equation (6) is the zero map. $\qquad \square$

In the special case that $\dim_{\mathbb{C}} H^1(N_K; sl(n, \mathbb{C})_{ad\,\alpha}) = n - 1$, we will prove integrability of all the tangent vectors and deduce that the corresponding point is a smooth point of the character variety.

In the next lemma, we denote by L_n the n–fold branched cover of Σ branched along the knot K.

Lemma 16. *Suppose $\alpha \colon \pi_1(N_K) \to \mathrm{SL}(n, \mathbb{C})$ is an irreducible metabelian representation. If $\dim_{\mathbb{C}} H^1(N_K; sl(n, \mathbb{C})_{ad\,\alpha}) = n - 1$, then $b_1(L_n) = 0$. Consequently α has finite image and is therefore conjugate to a unitary representation.*

Proof. Recall that we can assume that $\alpha = \alpha_{(n,\chi)}$ for some character $\chi \colon H_1(L_n) \to \mathbb{C}^*$. We denote by $\theta_1 \colon \pi_1(N_K) \to \mathrm{GL}(1, \mathbb{C})$ the trivial representation and we denote by $\alpha_n \colon \pi_1(N_K) \to \mathrm{Aut}(\mathbb{C}[\mathbb{Z}/n])$ the regular representation corresponding to the canonical projection map $\pi_1(N_K) \to \mathbb{Z} \to \mathbb{Z}/n$. By Lemma 6 we have the following isomorphism of representations:

$$ ad\,\alpha \oplus \theta_1 \cong \alpha_n \oplus \bigoplus_{i=1}^{n-1} \beta_{(n,\chi_i)}, $$

where $\chi_i, i = 1, \ldots, n - 1$ are characters. Note that θ_1, α_n and $ad\,\alpha$ are orthogonal (cf. Lemma 8). It follows that $\beta := \bigoplus_{i=1}^{n-1} \beta_{(n,\chi_i)} \colon \pi_1(N_K) \to \mathrm{GL}(n(n-1), \mathbb{C})$ is also

orthogonal. By Lemma 10 we now have

$$
\begin{aligned}
\dim_{\mathbb{C}} H^1(N_K; sl(n, \mathbb{C})_{ad\,\alpha}) &= \dim_{\mathbb{C}} H_1(N_K; \mathbb{C}[\mathbb{Z}/n]) - 1 + \dim_{\mathbb{C}} H^1(N_K; \mathbb{C}_\beta^{n(n-1)}) \\
&= b_1(L_n) + 1 - 1 + \dim_{\mathbb{C}} H^1(N_K; \mathbb{C}^{n(n-1)}) \\
&\geq b_1(L_n) + n - 1,
\end{aligned}
$$

The condition $\dim_{\mathbb{C}} H^1(N_K; sl(n, \mathbb{C})_{ad\,\alpha}) = n - 1$ now shows that $b_1(L_n) = 0$. Thus $H_1(L_n) = H_1(N_K; \mathbb{Z}[t^{\pm 1}])/(t^n - 1)$ is finite, and this implies α has finite image and is conjugate to a unitary representation. $\qquad\square$

Next we shall present a more general result which implies the first part of Theorem 2 as a special case. This result was proved previously by L. Abdelghani, M. Heusener, and H. Jebali in [AHJ07, Section 2], and we include a complete proof here for the reader's convenience. Before stating that result, we recall the following definition. A point $\xi \in X$ in an affine algebraic variety is called a *simple point* if it is contained in a unique algebraic component of X and is a smooth point of that component.

Theorem 17. *Suppose* $\alpha\colon \pi_1(N_K) \to \mathrm{SL}(n, \mathbb{C})$ *is an irreducible representation with* $\dim_{\mathbb{C}} H_1(N_K; sl(n, \mathbb{C})_{ad\,\alpha}) = n - 1$ *such that the image of the restriction* $\hat{\alpha}\colon \pi_1(\partial N_K) \to \mathrm{SL}(n, \mathbb{C})$ *contains a matrix with n distinct eigenvalues. Then ξ_α is a simple point in the character variety* $X_n(N_K)$.

Proof. As a first step, we will use the hypothesis to show that $\xi_{\hat{\alpha}}$ is a simple point in $X_n(\partial N_K)$, then we will apply the deformation obstruction machinery described in [HPS01] to conclude that ξ_α is a simple point of $X_n(N_K)$. By Luna's étale slice theorem [Lu73], we see that ξ_α is a simple point of $X_n(N_K)$ if and only if α is a simple point of $R_n(N_K)$, and similarly for $\hat{\alpha}$ and $\xi_{\hat{\alpha}}$. For α, this uses irreducibility, and for $\hat{\alpha}$, it follows by our hypothesis on the image of $\hat{\alpha}$.

Before beginning the argument, we introduce some notation. Given a finitely presented group π and representation $\alpha\colon \pi \to \mathrm{SL}(n, \mathbb{C})$, we use $H^*(\pi; sl(n, \mathbb{C})_{ad\,\alpha})$ to denote the cohomology of the group with coefficients in the π–module by $sl(n, \mathbb{C})_{ad\,\alpha}$.

In order to prove that $\hat{\alpha}$ is a simple point in $R_n(\partial N_K)$, we will compare the dimension of the cocycles $Z^1(\pi_1(\partial N_K); sl(n, \mathbb{C})_{ad\,\alpha})$ with the local dimension of $R_n(\partial N_K)$ at α, which is defined to be the maximal dimension of the irreducible components of $R_n(\partial N_K)$ containing α.

In [We64], Weil observed that there is a natural inclusion of the Zariski tangent space $T_\alpha^{\mathrm{Zar}}(R_n(\pi)) \hookrightarrow Z^1(\pi; sl(n, \mathbb{C})_{ad\,\alpha})$ into the space of cocycles, and we will combine this observation with computations of the twisted cohomology of $\pi_1(\partial N_K)$ and $\pi_1(N_K)$ to complete the proof.

Because ∂N_K is a $K(\mathbb{Z} \oplus \mathbb{Z}, 1)$, we have isomorphisms

$$
H^*(\partial N_K; sl(n, \mathbb{C})_{ad\,\alpha}) \longrightarrow H^*(\pi_1(\partial N_K); sl(n, \mathbb{C})_{ad\,\alpha}),
$$

and as explained in [HP05, Lemma 3.1], the inclusion $N_K \hookrightarrow K(\pi_1(N_K), 1)$ induces maps

$$
H^i(\pi_1(N_K); sl(n, \mathbb{C})_{ad\,\alpha}) \longrightarrow H^i(N_K; sl(n, \mathbb{C})_{ad\,\alpha})
$$

that are isomorphisms when $i = 0$ and 1 and injective when $i = 2$.

We equip the torus ∂N_K with the standard CW–decomposition consisting of one 0–cell, two 1–cells and one 2–cell. It is straightforward to verify that the spaces of twisted 1-coboundaries and 1-cocycles satisfy

$$\dim_{\mathbb{C}} B^1(\partial N_K; sl(n, \mathbb{C})_{ad\,\hat{\alpha}}) = n^2 - 1 - (n-1) = n^2 - n$$

and

$$\dim_{\mathbb{C}} Z^1(\partial N_K; sl(n, \mathbb{C})_{ad\,\hat{\alpha}}) = 2(n-1) + n^2 - n = n^2 + n - 2.$$

Now observe that $\hat{\alpha}$ sits on an $n^2 + n - 2$ dimensional component, hence its local dimension is

$$\dim_{\alpha} R_n(\partial N_K) = n^2 + n - 2.$$

For arbitrary $\alpha \in R_n(\partial N_K)$, we have

$$\dim_{\alpha} R_n(\partial N_K) \leq \dim_{\mathbb{C}} T_{\alpha}^{\mathrm{Zar}}(R_n(N_K)) \leq \dim_{\mathbb{C}} Z^1(\partial N_K; sl(n, \mathbb{C})_{ad\,\hat{\alpha}}).$$

In our case, we have equality throughout, and it follows that $\hat{\alpha}$ lies on a unique irreducible component of $R_n(\partial N_K)$ and is a smooth point of that component (see [Sh95, §2, Theorem 6]). This completes the proof that α is a simple point of $R_n(\partial N_K)$.

In order to prove that ξ_{α} is a simple point of $X_n(N_K)$, we consider again the long exact sequence (5) in cohomology associated with the pair $(N_K, \partial N_K)$ and note that irreducibility of α implies that $H^0(N_K; sl(n, \mathbb{C})_{ad\,\alpha}) = 0$. It follows from Lemmas 7, 8 and Poincaré duality that $H^3(N_K, \partial N_K; sl(n, \mathbb{C})_{ad\,\alpha}) = 0$. Further, by hypothesis we have $H^1(N_K; sl(n, \mathbb{C})_{ad\,\alpha}) = \mathbb{C}^{n-1}$, and so we also have $H^2(N_K, \partial N_K; sl(n, \mathbb{C})_{ad\,\alpha}) = \mathbb{C}^{n-1}$ by Poincaré duality.

Since $H^1(\partial N_K; sl(n, \mathbb{C})_{ad\,\hat{\alpha}}) = \mathbb{C}^{2(n-1)}$, it follows immediately that the middle row of (5)

$$0 \to H^1(N_K; sl(n, \mathbb{C})_{ad\,\alpha}) \to H^1(\partial N_K; sl(n, \mathbb{C})_{ad\,\hat{\alpha}}) \to H^2(N_K, \partial N_K; sl(n, \mathbb{C})_{ad\,\alpha}) \to 0$$

is short exact. Thus $j^1 = 0$ and $j^2 = 0$, and further i^1 is injective and i^2 is an isomorphism.

We now use a powerful method for deforming representations that involves constructing formal deformations by computing the vanishing of all obstructions. This technique was first developed in [HPS01] for SL$(2, \mathbb{C})$ and later extended to other situations in [HP05, AHJ07]. Here we explain how to carry over their arguments to the SL$(n, \mathbb{C})$ case, and we use that approach to prove integrability of all elements in $Z^1(\pi_1(N_K); sl(n, \mathbb{C})_{ad\,\alpha})$ under the present circumstances.

We begin by describing the general framework of formal deformation theory and the related obstructions. Suppose π is a finitely presented group and $\alpha \colon \pi \to \mathrm{SL}(n, \mathbb{C})$ is a representation. A *formal deformation* of α is a homomorphism $\alpha_{\infty} \colon \pi \to \mathrm{SL}(n, \mathbb{C}[[t]])$ given by

$$\alpha_{\infty}(g) = \exp\left(\sum_{i=1}^{\infty} t^i a_i(g)\right) \alpha(g),$$

such that $p_0(\alpha_\infty) = \alpha$, where $p_0 \colon \mathrm{SL}(n, \mathbb{C}[[t]]) \to \mathrm{SL}(n, \mathbb{C})$ is the homomorphism given by setting $t = 0$ and where $a_i \colon \pi \to sl(n, \mathbb{C})_{ad\,\alpha}, i = 1, \ldots,$ are 1-cochains with twisted coefficients. By [HPS01, Lemma 3.3], it follows that α_∞ is a homomorphism if and only if $a_1 \in Z^1(\pi; sl(n, \mathbb{C})_{ad\,\alpha})$ is a cocycle, and we call an element $a \in Z^1(\pi; sl(n, \mathbb{C})_{ad\,\alpha})$ *formally integrable* if there is a formal deformation with leading term $a_1 = a$.

Let $a_1, \ldots, a_k \in C^1(\pi; sl(n, \mathbb{C})_{ad\,\alpha})$ be cochains such that

$$\alpha_k(g) = \exp\left(\sum_{i=1}^{k} t^i a_i(g)\right) \alpha(g)$$

is a homomorphism into $\mathrm{SL}(n, \mathbb{C}[[t]])$ modulo t^{k+1}. Here, α_k is called a formal deformation of order k, and in this case by [HPS01, Proposition 3.1] there exists an obstruction class $\omega_{k+1} := \omega_{k+1}^{(a_1,\ldots,a_k)} \in H^2(\pi; sl(n, \mathbb{C})_{ad\,\alpha})$ with the following properties:

(1) There is a cochain $a_{k+1} \colon \pi \to sl(n, \mathbb{C})$ such that

$$\alpha_{k+1}(g) = \exp\left(\sum_{i=1}^{k+1} t^i a_i(g)\right) \alpha(g)$$

is a homomorphism modulo t^{k+2} if and only if $\omega_{k+1} = 0$.
(2) The obstruction ω_{k+1} is natural, i.e. if $\varphi \colon \pi' \to \pi$ is a homomorphism then $\varphi^* \omega_k := \alpha_k \circ \varphi$ is also a homomorphism modulo t^{k+1} and $\varphi^*(\omega_{k+1}^{(a_1,\ldots,a_k)}) = \omega_{k+1}^{(\varphi^* a_1,\ldots,\varphi^* a_k)}$.

Lemma 18. *Let $\alpha \colon \pi_1(N_K) \to \mathrm{SL}(n, \mathbb{C})$ be an irreducible representation such that $\dim_{\mathbb{C}} H^1(N_K; sl(n, \mathbb{C})_{ad\,\alpha}) = n - 1$. If the image of the restriction $\hat{\alpha} \colon \pi_1(\partial N_K) \to \mathrm{SL}(n, \mathbb{C})$ contains an element with n distinct eigenvalues, then every cocycle $a \in Z^1(\pi_1(N_K); sl(n, \mathbb{C})_{ad\,\alpha})$ is integrable.*

Proof. Consider first the commutative diagram:

$$
\begin{array}{ccc}
H^2(\pi_1(N_K); sl(n, \mathbb{C})_{ad\,\alpha}) & \xrightarrow{\;i^*\;} & H^2(\pi_1(\partial N_K); sl(n, \mathbb{C})_{ad\,\hat{\alpha}}) \\
\downarrow & & \downarrow{\scriptstyle\cong} \\
H^2(N_K; sl(n, \mathbb{C})_{ad\,\alpha}) & \xrightarrow{\;\cong\;} & H^2(\partial N_K; sl(n, \mathbb{C})_{ad\,\hat{\alpha}}).
\end{array}
$$

Here, the horizontal isomorphism on the bottom follows by consideration of the long exact sequence (5), and the vertical isomorphism on the right follows since ∂N_K is a $K(\mathbb{Z} \oplus \mathbb{Z}, 1)$. Further, we know the vertical map on the left is an injection by [HP05, Lemma 3.3], and this shows i^* is an injection.

We now prove that every element $a \in Z^1(\pi_1(N_K); sl(n, \mathbb{C})_{ad\,\alpha})$ is integrable. Suppose (by induction) that $a_1, \ldots, a_k \in C^1(\pi; sl(n, \mathbb{C})_{ad\,\alpha})$ are given so that

$$\alpha_k(g) = \exp\left(\sum_{i=1}^k t^i a_i(g)\right) \alpha(g)$$

is a homomorphism modulo t^{k+1}. Then the restriction $\hat{\alpha}_k \colon \pi_1(\partial N_K) \to \mathrm{SL}(n, \mathbb{C}[[t]])$ is also a formal deformation of order k. On the other hand, $\hat{\alpha}_k$ is a smooth point of $R_n(\partial N_K)$, hence by [HPS01, Lemma 3.7], $\hat{\alpha}_k$ extends to a formal deformation of order $k + 1$. Therefore

$$0 = \omega_{k+1}^{(i^*a_1, \ldots, i^*a_k)} = i^* \omega_{k+1}^{(a_1, \ldots, a_k)}.$$

As i^* is injective, the obstruction vanishes. $\square$

We are now ready to conclude the proof of Theorem 17. Using Lemma 18, we see that all cocycles in $Z^1(\pi_1(N_K); sl(n, \mathbb{C})_{ad\,\alpha})$ are integrable. Applying Artin's theorem [Ar68], we obtain from a formal deformation of α a convergent deformation (see [HPS01, Lemma 3.6]). Thus α is a smooth point of $R_n(N_K)$ with local dimension $\dim_\alpha R_n(N_K) = n - 1$. It follows that α is a simple point of $R_n(N_K)$ and this together with irreducibility of α imply that ξ_α is a simple point of $X_n(N_K)$. $\square$

The proof of Theorem 17 is easily adapted to the $SU(n)$ setting to prove the following statement, where we use $X_{SU(n)}(N)$ to denote the character variety of $SU(n)$ representations of $\pi_1(N)$.

Theorem 19. *Suppose $\alpha \colon \pi_1(N_K) \to SU(n)$ is an irreducible representation with $\dim_\mathbb{R} H_1(N_K; su(n)_{ad\,\alpha}) = n - 1$ such that the image of $\hat{\alpha} \colon \pi_1(\partial N_K) \to SU(n)$ contains a matrix with n distinct eigenvalues. Then ξ_α is a simple point in the character variety $X_{SU(n)}(N_K)$.*

We now complete the proof of Theorem 2. If α is an irreducible metabelian representation with $\dim_\mathbb{C} H^1(N_K; sl(n, \mathbb{C})_{ad\,\alpha}) = n - 1$, then Lemma 16 applies and shows that α has finite image and hence is conjugate to a unitary representation. Since $\alpha(\mu)$ has n distinct eigenvalues, Theorem 17 applies and gives rise to a smooth complex $(n - 1)$–dimensional family of $\mathrm{SL}(n, \mathbb{C})$ characters near $\xi_\alpha \in X_n(N_K)$.

Note that, since α is unitary and $H^1(N_K; sl(n, \mathbb{C})_{ad\,\alpha})$ is the complexification of $H^1(N_K; su(n)_{ad\,\alpha})$, we see that

$$\dim_\mathbb{R} H^1(N_K; su(n)_{ad\,\alpha}) = \dim_\mathbb{C} H^1(N_K; sl(n, \mathbb{C})_{ad\,\alpha}).$$

Thus Theorem 19 applies and gives rise to a smooth real $(n - 1)$–dimensional family of irreducible characters near $\xi_\alpha \in X_{SU(n)}(N_K)$.

Note that Lemma 16 shows that $b_1(L_n) = 0$, and thus every irreducible metabelian representation $\beta \colon \pi_1(N_K) \to \mathrm{SL}(n, \mathbb{C})$ factors through a finite group. In particular, this shows that up to conjugacy there are only finitely many irreducible metabelian $\mathrm{SL}(n, \mathbb{C})$ representations, and their characters give rise to a finite collection of points in

the character variety $X_n^*(N_K)$. It follows that we can take either of the two deformation families of conjugacy classes of irreducible representations so that ξ_α is the unique metabelian representation within the family.

4.3. **Proof of Theorem 3.**

Proof of Theorem 3. Let K be a knot, $\alpha\colon \pi_1(N_K) \to \mathrm{SL}(n,\mathbb{C})$ an irreducible metabelian representation, and $\varphi\colon \pi_1(N_K) \to \mathbb{Z}/n \ltimes H$ a homomorphism such that α factors through φ and with H is finite. Set $k = |H|$.

We first consider the cover $p\colon \widetilde{N}_\varphi \to N_K$ corresponding to φ. Note that there exist precisely $k = |H|$ characters $H \to U(1)$. We denote this set by $\{\sigma_1, \dots, \sigma_k\}$, where we assume that σ_1 is the trivial character. It is not difficult to see that the representation $\sigma_1 \oplus \cdots \oplus \sigma_k\colon H \to \mathrm{Aut}(\mathbb{C}^k)$ is isomorphic to the regular representation $H \to \mathrm{Aut}(\mathbb{C}[H])$. We denote the representation $\pi_1(N_K) \to \mathrm{Aut}(\mathbb{C}[\mathbb{Z}/n \ltimes H])$ by φ as well. Then it is straightforward to verify that

$$\varphi \cong \bigoplus_{i=1}^{k} \beta_{(n,\sigma_i)}.$$

In particular, setting $V = \mathbb{C}^{kn}$ and $U = \mathbb{C}^n$, we have

$$b_1(\widetilde{N}_\varphi) = b_1(N;V_\varphi) = \sum_{i=1}^{k} b_1(N;U_{\beta_{(n,\sigma_i)}}).$$

Note that $\beta_{(n,\sigma_i)}$ is a unitary representation. It now follows immediately from Lemma 10 that $b_1(\widetilde{N}_\varphi) \geq k$. Furthermore, if $b_1(\widetilde{N}_\varphi) = k$ then it follows that $b_1(N;U_{\beta_{(n,\sigma_i)}}) = 1$ for any i. Statement (i) now follows immediately from Lemma 6.

We now turn to the proof of (ii). We write $T = \partial N_K$. Note that the image of the restriction $\hat{\varphi}\colon \pi_1(T) \to \mathbb{Z}/n \ltimes H$ has order n. In particular the preimage of T under the covering $p\colon \widetilde{N}_\varphi \to N_K$ has $k = |H|$ components. We denote the components by $T_1, \dots, T_k$. Note that in each T_i there exist simple closed curves μ_i and λ_i such that $p|_{\mu_i}$ restricts to an n–fold cover of the meridian $\mu \subset T$ and such that $p|_{\lambda_i}$ restricts to a homeomorphism with the longitude λ of T. Note that μ_i, λ_i form a basis for $H_1(T_i)$.

We now denote by $\widehat{\Sigma}_\varphi$ the result of gluing k solid tori $S_1, \dots, S_k$ to the boundary of $\widetilde{N}_\varphi$ such that each μ_i bounds a disk in S_i. The projection map $p\colon \widetilde{N}_\varphi \to N_K$ now extends in a canonical way to a covering map $\widehat{\Sigma}_\varphi \to \Sigma$, branched over K.

We finally turn to the proof of (iii). Consider the following Mayer–Vietoris sequence:

$$\bigoplus_{i=1}^{k} H_1(T_i) \to \bigoplus_{i=1}^{k} H_1(S_i) \ \oplus \ H_1(\widetilde{N}_\varphi) \to H_1(\widehat{\Sigma}_\varphi) \to 0.$$

It follows immediately that

$$b_1(\widehat{\Sigma}_\varphi) \geq k + b_1(\widetilde{N}_\varphi) - 2k = b_1(\widetilde{N}_\varphi) - k.$$

In particular if $b_1(\widehat{\Sigma}_\varphi) = 0$, then $b_1(\widetilde{N}_\varphi) \leq k$. We can therefore now apply (i) to deduce that (iii) holds. $\qquad\square$

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Department of Mathematics, McMaster University, Hamilton, Ontario
E-mail address: boden@mcmaster.ca

University of Warwick, Coventry, UK
E-mail address: sfriedl@gmail.com